Calculating matrix definiteness

$$M = \begin{bmatrix} a & 1 & b \\ 1 & -1 & 0 \\ b & 0 & -2 \end{bmatrix}$$

Determine the values of a and b for which the matrix M is classified as negative definite, negative semidefinite, positive definite, positive semidefinite, and indefinite. (Note that there may not exist any values of a and b that fulfill some of these classifications.)

Solution

If the matrix is symmetric, let D_k denote its leading principal minors for $k = 1, \ldots, n$. Then, the following conditions apply:

- Matrix A is termed positive definite if and only if $D_k > 0$ holds for each $k = 1, \ldots, n$.
- Matrix A is referred to as negative definite if and only if $(-1)^k D_k > 0$ for all $k = 1, \ldots, n$. This condition is satisfied if the leading principal minors alternate in sign, commencing with a negative value for D_1.

For semidefiniteness of a symmetric matrix the following criteria are applied:

- A is a positive semidefinite if and only if every principal minor of A is nonnegative.
- A is considered negative semidefinite if and only if for each $k = 1, \ldots, n$, every k-th order principal minor of A is nonpositive when k is odd, and nonnegative when k is even.

Let's calculate the leading principal minors:

$$D_1 = a$$

$$D_2 = -1 - a$$

$$D_3 = b \cdot (-b) - 2 \cdot (-a - 1) = -b^2 + 2a + 2$$

Therefore, the matrix is negative definite if and only if

- $D_1 = a < 0$
- $D_2 = -1 - a > 0$
- $D_3 = b^2 + 2a + 2 < 0$

In conclusion: $a < -1$ and $2a + 2 + b^2 < 0$.

The matrix is negative semidefinite if and only if:

- $M_{1,1} = 2 \geq 0$
- $M_{2,2} = -2 - b^2 \leq 0$
- $M_{3,3} = -a - 1 \geq 0$
- $det(M) = 2a + b^2 + 2 \leq 0$

For the given matrix, it is not classified as positive definite or positive semidefinite since two of the first-order principal minors are negative.